TIGHT FRAME COMPLETIONS WITH PRESCRIBED NORMS.

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Abstract. Let $\mathcal{H}$ be a finite dimensional (real or complex) Hilbert space and let $\{a_i\}_{i=1}^{\infty}$ be a non-increasing sequence of positive numbers. Given a finite sequence of vectors $\mathcal{F} = \{f_i\}_{i=1}^{p}$ in $\mathcal{H}$ we find necessary and sufficient conditions for the existence of $r \in \mathbb{N} \cup \{\infty\}$ and a Bessel sequence $\mathcal{G} = \{g_i\}_{i=1}^{r}$ in $\mathcal{H}$ such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame for $\mathcal{H}$ and $\|g_i\|^2 = a_i$ for $1 \le i \le r$. Moreover, in this case we compute the minimum $r \in \mathbb{N} \cup \{\infty\}$ with this property. Using recent results on the Schur-Horn theorem, we also obtain a not so optimal but algorithmic computable (in a finite numbers of steps) tight completion sequence $\mathcal{G}$.

Keywords: frame, tight frame completion, majorization.

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1. Introduction.

In recent years, the study of frames in finite dimensional Hilbert spaces has been motivated by a large variety of applications, such as signal processing, multiple antenna coding, perfect reconstruction filter banks, and Sampling Theory.

Some particular frames, called *tight* frames, are of special interest since they allow simple reconstruction formulas. For practical purposes, is often useful to obtain tight frames with some extra "structure", for example with the norms of its elements prescribed (controlled) in advance.

In [4] D. Feng, L. Wang and Y. Wang considered the problem of computing tight completions of a given set of vectors. More explicitly, given a finite sequence $\mathcal{F} = \{f_i\}_{i=1}^{p}$ of vectors in $\mathcal{H}$, how many vectors we have to add in order to obtain a tight frame, and how to find those vectors?. Theorem 1.1 in [4] provides a complete answer to this question. But when the norms of the additional vectors are required to be one (with the initial set of given vectors of norm one) the authors obtained a lower bound for the number of unit norm vectors we have to add ([4],Theorem 1.2); but they showed that their lower bound is not sharp in some cases.

In this note, we calculate the minimum number of vectors we have to add to $\mathcal{F}$ to obtain a *tight completion*. Moreover, we do not require the vectors to be of norm one; we look for tight completions with sequences of vectors whose squared norms are prescribed by a non-increasing sequence of positive numbers.

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Note that this problem may not have a positive solution for a given set of initial vectors and a fixed sequence of "prescribed norms". Therefore we first find conditions for such a tight frame completion to exist. The main tool used here is Theorem 2.3, which relates the squared norms of the vectors in a Bessel sequence with the spectrum of its frame operator.

In order to state our main results, we fix some notation used throughout the paper. Let $\mathcal{H}$ be a real or complex finite dimensional vector space with $\dim \mathcal{H} = n \in \mathbb{N}$. Let $\mathcal{F} = \{f_i\}_{i=1}^p \subseteq \mathcal{H}$ be a finite sequence with frame operator $S^{\mathcal{F}}$ whose eigenvalues (counted with multiplicity) are $\lambda_1 \geq \ldots \geq \lambda_n$, and let $\mathbf{a} = \{a_i\}_{i\in\mathbb{N}}$ be a non-increasing sequence of positive real numbers. Finally, let $\alpha = \operatorname{tr}(S^{\mathcal{F}})$.

Theorem (A)**.** *Given $r \in \mathbb{N}$, there exists $\mathcal{G} = \{g_i\}_{i=1}^r \subseteq \mathcal{H}$ such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame if and only if $\frac{1}{n}\left(\sum_{i=1}^r a_i + \alpha\right) \geq \lambda_1$ and*

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \geq \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1}),\ 1 \leq k \leq \min\{n, r\}. \tag{1}$$

On the other hand, there exists an infinite Bessel sequence $\mathcal{G} = \{g_i\}_{i=1}^\infty$ in $\mathcal{H}$ such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame if and only if $\{a_i\}_{i=1}^\infty \in \ell^1(\mathbb{N})$, $\frac{1}{n}\left(\sum_{i=1}^\infty a_i + \alpha\right) \geq \lambda_1$ and

$$\frac{1}{n}\left(\sum_{i=1}^{\infty} a_i + \alpha\right) \geq \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1}),\ 1 \leq k \leq n. \tag{2}$$

So from Theorem A we get necessary and sufficient conditions for the existence of a sequence $\mathcal{G} = \{g_i\}_{i=1}^r$, for some $r \in \mathbb{N} \cup \{\infty\}$, with $\|g_i\|^2 = a_i$, and such that $\mathcal{F} \cup \mathcal{G}$ is a tight frame (for some suitable constant). If such a completion exists we say that $\mathcal{F}$ is $(\mathbf{a}, r)$-completable. In case $\mathcal{F}$ is $(\mathbf{a}, r)$-completable, we are then interested in computing the minimum number r_0 of vectors we have to add. In order to state our next result we introduce the following numbers: let $c_0 = \lambda_1$ and for $1 \leq k \leq n$ let

$$c_k = \max\left(c_{k-1},\ \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1})\right). \tag{3}$$

Theorem (B)**.** *Assume that $\mathcal{F}$ is $(\mathbf{a}, r)$-completable for some $r \in \mathbb{N} \cup \{\infty\}$ and let $r_0 \in \mathbb{N} \cup \{\infty\}$ be the minimum such that $\mathcal{F}$ is $(\mathbf{a}, r_0)$-completable.*

Then

Case 1: *$r_0 < n$ if and only if $c_{r_0} = \frac{1}{n}\left(\sum_{i=1}^{r_0} a_i + \alpha\right)$.*

Case2: *$n \leq r_0 < \infty$ if and only if $c_k \neq \frac{1}{n}\left(\sum_{i=1}^{k} a_i + \alpha\right)$ $\forall\, 1 \leq k \leq n-1$ and r_0 is the minimum such that $c_n \leq \frac{1}{n}\left(\sum_{i=1}^{r_0} a_i + \alpha\right)$.*

Case 3: *$r_0 = \infty$ if and only if $c_k \neq \frac{1}{n}\left(\sum_{i=1}^{k} a_i + \alpha\right)$ for all $1 \leq k \leq n-1$ and $c_n = \frac{1}{n}\left(\sum_{i=1}^{\infty} a_i + \alpha\right)$.*

We should remark that although Theorems A and B are of practical interest, they are not efficiently (fast) algorithmic implementable in a computer (see the discussion

at the beginning of Section 5). In Section 5 we deal with the problem of finding a not so optimal but efficiently algorithmic computable finite tight completion as follows:

Theorem (C)**.** *Assume that* $\mathbf{a}$ *is a divergent sequence. Let* $d \in \mathbb{R}$ *be an algorithmic computable upper bound for* $\|S^{\mathcal{F}}\|$ *and let* $c = \max(d+1, d+a_1)$*. If* $r \in \mathbb{N}$ *is such that*

$$\sum_{i=1}^{r-1} a_i < c \cdot n - \operatorname{tr}(S^{\mathcal{F}}) \leq \sum_{i=1}^{r} a_i$$

then there exists an algorithmic computable sequence $\mathcal{G} = \{g_i\}_{i=1}^r$ *such that* $\mathcal{F} \cup \mathcal{G}$ *is a tight frame and such that* $\|g_i\|^2 = a_i$ *for* $1 \leq i \leq r$.

We also consider particular cases of Theorems A and B when $a_i = 1$ for $i \geq 1$.

2. Preliminaries on frames and majorization

Throughout the paper, $\mathcal{H}$ will be a finite dimensional (real or complex) Hilbert space with $\dim \mathcal{H} = n \in \mathbb{N}$ and $L(\mathcal{H})^+$ will denote the cone of bounded positive semi-definite operators on $\mathcal{H}$. Given $m \in \mathbb{N} \cup \{\infty\}$, a sequence $\mathcal{F} = \{f_i\}_{i=1}^m \subset \mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist numbers $a, b > 0$ such that, for every $f \in \mathcal{H}$,

$$a\,\|f\|^2 \leq \sum_{i=1}^{m} |\,\langle f, f_i\rangle\,|^2 \leq b\,\|f\|^2 \tag{4}$$

The optimal constants in (4) are called the *frame bounds*. If the frame bounds a, b coincide, the frame is called *a-tight* (or simply tight). Finally, tight frames with all its elements having the same norm are called *equal norm tight frames*.

The sequence $\mathcal{F}$ is *Bessel* if there exists $b > 0$ such that the upper bound condition in (4) is satisfied. Given a Bessel sequence $\mathcal{F}$, we define its *frame operator* by

$$S^{\mathcal{F}} f = \sum_{i=1}^{m} \langle f, f_i\rangle\, f_i. \tag{5}$$

It is easy to see that $S^{\mathcal{F}}$ is a positive semi-definite bounded operator on $\mathcal{H}$. Moreover, $\mathcal{F}$ is a frame if and only if its frame operator $S^{\mathcal{F}}$ is invertible. Indeed, the optimal frame bounds a, b in (4) are respectively $\lambda_{\min}(S^{\mathcal{F}})$ and $\lambda_{\max}(S^{\mathcal{F}})$, the minimum and maximum eigenvalues of $S^{\mathcal{F}}$. In particular, a frame $\mathcal{F}$ is a-tight if and only if $S^{\mathcal{F}} = aI$. For an introduction to the theory of frames and related topics see the books [6, 10].

Given a Bessel sequence $\mathcal{F}$, there is a close relationship between the norms of its elements and the spectrum of $S^{\mathcal{F}}$ that can be expressed in terms of majorization (see [1] for details). First, we introduce some definitions. We say that a sequence $\{a_i\}_{i=1}^m$ is *summable* if $m \in \mathbb{N}$, or if $m = \infty$ and $\{a_i\}_{i=1}^\infty \in \ell^1(\mathbb{N})$.

Definition 2.1. *Let* $\mathbf{a} = \{a_i\}_{i=1}^m$*,* $\boldsymbol{b} = \{b_i\}_{i=1}^s$ *be non-increasing summable sequences of non-negative numbers, with* $s, m \in \mathbb{N} \cup \{\infty\}$*, and let* $t = \min\{s, m\}$*. We*

say that $\mathbf{b}$ *majorizes* $\mathbf{a}$*, noted* $\mathbf{b} \succ \mathbf{a}$*, if*

$$\sum_{i=1}^{j} b_i \geq \sum_{i=1}^{j} a_i \quad \text{for } 1 \leq j \leq t \quad \text{and} \quad \sum_{i=1}^{s} b_i = \sum_{i=1}^{m} a_i. \tag{6}$$

If $m = s \in \mathbb{N}$ in Definition 2.1 then this notion coincides with the usual vector majorization in $\mathbb{R}^m$ between vectors with non-negative entries which are arranged in non-increasing order (see [8]).

On the other hand, as an immediate consequence of Definition 2.1 we see that if $s \in \mathbb{N}$, and then $\mathbf{a} \prec \mathbf{b}$ if and only if $\mathbf{a} \prec (\mathbf{b}, 0_n)$ for every $n \in \mathbb{N}$, where $(\mathbf{b}, 0_n) \in \mathbb{R}^{s+n}$, and similarly $(\mathbf{a}, 0_n) \prec \mathbf{b}$ if $m \in \mathbb{M}$.

Remark 2.2. Let $\mathbf{a}$, $\mathbf{b}$ be as in Definition 2.1, with $\mathbf{b} \succ \mathbf{a}$ and $m < s$. Then $b_i = 0$ for $m + 1 \leq i \leq s$, since

$$\sum_{j=1}^{m} a_j = \sum_{j=1}^{s} b_j \geq b_i + \sum_{j=1}^{m} b_j \geq \sum_{j=1}^{m} a_j$$

which implies that $b_i = 0$ since $\mathbf{b}$ has non-negative entries.

Now we can state the frame version of the Schur-Horn theorem, which we shall need in the sequel.

Theorem 2.3. *Let* $\mathbf{a} = \{a_i\}_{i=1}^m$ *be a non-increasing sequence of positive numbers and let* $S \in L(\mathcal{H})^+$ *with eigenvalues (counted with multiplicity and arranged in non-increasing order)* $\mathbf{\Lambda} = \{\lambda_j\}_{j=1}^n$*. Then the following statements are equivalent:*

(1) $\mathbf{a} \prec \mathbf{\Lambda}$.
(2) *There exists a Bessel sequence* $\mathcal{G} = \{g_i\}_{i=1}^m \subset \mathcal{H}$ *such that* $\|g_i\|^2 = a_i$ *for* $1 \leq i \leq m$ *and* $S^{\mathcal{G}} = S$.

Proof. If we assume that $S > 0$ then the case when $m \in \mathbb{N}$ is Theorem 4.6 in [1], while the case when $m = \infty$ is Theorem 4.7 in [1]. If the spectrum of S has zeros (note that this is the case whenever $m < n$) we can reduce to the invertible case, restricting S to the orthogonal complement of $\ker S$. □

Remark 2.4. We have now a way to look at the problem of tight completions of a given set of vectors: a set $\mathcal{F} = \{f_i\}_{i=1}^p$ has a c-tight completion $\mathcal{G} = \{g_i\}_{i=1}^m$ if and only if $S^{\mathcal{G}} = cI - S^{\mathcal{F}}$. Thus, by Theorem 2.3,this is equivalently to the fact that the squared norms of $\{g_i\}_{i=1}^m$ are majorized by the non-increasing sequence $c - \lambda_n \geq c - \lambda_{n-1} \geq \ldots \geq c - \lambda_1$, where λ_i are the eigenvalues of $S^{\mathcal{F}}$ (counted with multiplicity and rearranged in decreasing order).

3. Completing a Bessel sequence to a tight frame with prescribed norms

Definition 3.1. *We say that* $\mathcal{F}$ *is* $(\mathbf{a}, r)$**-completable** *if there exists* $r \in \mathbb{N} \cup \{\infty\}$ *and a Bessel sequence* $\mathcal{G} = \{g_i\}_{i=1}^r \subset \mathcal{H}$*, with* $\|g_i\|^2 = a_i$ *for* $1 \leq i \leq r$*, and such*

that $\mathcal{F}\cup\mathcal{G}$ is a tight frame. We say that $\mathcal{G}=\{g_i\}_{i=1}^r$ is an **$(\mathbf{a},r)$-completion** *of $\mathcal{F}$.*

Remark 3.2. If $\mathcal{G}=\{g_i\}_{i=1}^r$ is an $(\mathbf{a},r)$-completion of $\mathcal{F}$ then the frame bound $c\in\mathbb{R}$ for $\mathcal{F}\cup\mathcal{G}$ is determined by the number $r\in\mathbb{N}\cup\{\infty\}$ and the norms of the vectors of $\mathcal{F}$. In fact $\operatorname{tr}(S^{\mathcal{F}\cup\mathcal{G}})=nc$, and simple computations show that

$$\operatorname{tr}(S^{\mathcal{F}\cup\mathcal{G}})=\sum_{i=1}^p\|f_i\|^2+\sum_{i=1}^r a_i$$

so we have that $c=\frac{1}{n}(\sum_{i=1}^r a_i+\operatorname{tr}(S^{\mathcal{F}}))$. In particular, if $r=\infty$ then $\mathbf{a}$ is summable.

For the sake of clarity in the exposition, in what follows we consider separately the cases where $\mathcal{F}$ is $(\mathbf{a},r)$-completable for some $r\in\mathbb{N}$ and the case $r=\infty$, although there is no substantial difference in the arguments involved.

3.1. **Completing with a finite number of vectors.**

Theorem 3.3. *Let $r\in\mathbb{N}$. Then $\mathcal{F}$ is $(\mathbf{a},r)$-completable if and only if $\frac{1}{n}\left(\sum_{i=1}^r a_i+\alpha\right)\geq\lambda_1$ and*

$$\frac{1}{n}\left(\sum_{i=1}^r a_i+\alpha\right)\geq\frac{1}{k}\sum_{i=1}^k(a_i+\lambda_{n-i+1}),\ 1\leq k\leq\min\{n,r\}. \tag{7}$$

Proof. Assume that there exists $r\in\mathbb{N}$ and a finite sequence $\mathcal{G}=\{g_i\}_{i=1}^r$ such that $S^{\mathcal{F}\cup\mathcal{G}}=S^{\mathcal{F}}+S^{\mathcal{G}}=cI$ and $\|g_i\|^2=a_i$ for $1\leq i\leq r$. Then $cI-S^{\mathcal{F}}=S^{\mathcal{G}}\geq 0$; in particular we have $c\geq\|S\|=\lambda_1$. On the other hand, we see that the eigenvalues of $S^{\mathcal{G}}$ arranged in non-increasing order are $c-\lambda_n\geq\ldots\geq c-\lambda_1\geq 0$. By Theorem 2.3 we have

$$(c-\lambda_n,c-\lambda_{n-1},\ldots,c-\lambda_1)\succ(a_1,\ldots,a_r). \tag{8}$$

Then, by Definition 2.1 we see that $\frac{1}{n}\left(\sum_{i=1}^r a_i+\alpha\right)\geq\lambda_1$ and (7) hold, using that $c=\frac{1}{n}(\sum_{i=1}^r a_i+\alpha)$ by Remark 3.2.

Conversely assume that $\frac{1}{n}\left(\sum_{i=1}^r a_i+\alpha\right)\geq\lambda_1$ and (7) hold for $r\in\mathbb{N}$. Set $c=\frac{1}{n}(\sum_{i=1}^r a_i+\alpha)$ and note that the spectrum of the positive operator $cI-S^{\mathcal{F}}$, $(c-\lambda_n,c-\lambda_{n-1},\ldots,c-\lambda_1)$, majorizes (in the sense of Definition 2.1) $\{a_i\}_{i=1}^r$. By Theorem 2.3 we conclude that there exists a finite sequence $\mathcal{G}=\{g_i\}_{i=1}^r$ with $S^{\mathcal{G}}=cI-S^{\mathcal{F}}$ and $\|g_i\|^2=a_i$ for $1\leq i\leq r$ and we are done. $\square$

Remark 3.4. As a consequence of Theorem 3.3 we see that if $\sum_{i=1}^\infty a_i$ diverges, then every set of initial vectors $\mathcal{F}$ is $(\mathbf{a},r)$-completable for some $r\in\mathbb{N}$. We shall consider this in section 4 where we have $a_i=1$ for $i\in\mathbb{N}$.

By inspection of the proof of Theorem 3.3 and Remark 2.2, we have the following corollaries.

Corollary 3.5. *Using the notations of Theorem 3.3, $\mathcal{F}$ is $(\mathbf{a}, r)$-completable with $r < n$ if and only if, for $1 \le i \le n-r$ and $1 \le k \le r$,*

$$\lambda_i = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right), \quad and \;\; \lambda_1 \ge \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1}). \tag{9}$$

Corollary 3.6. *Let $\mathcal{F}$ be $(\mathbf{a}, r)$- completable for some $r \in \mathbb{N}$. Then*

(1) *if $r < n$ then $\mathcal{F}$ is not $(\mathbf{a}, k)$-completable for any $k < n$ other than r,*
(2) *if $r \ge n$ then $\mathcal{F}$ is $(\mathbf{a}, k)$-completable for every $k \in \mathbb{N}$ with $k \ge r$.*

The next result gives different equivalent conditions for a sequence $\mathbf{a}$ and vectors $\mathcal{F}$ in order to be $(\mathbf{a}, r)$-completable for some $r \in \mathbb{N}$. First, we define inductively the following numbers: let $c_0 = \lambda_1$ and for $1 \le k \le n$ let

$$c_k = \max\left(c_{k-1},\ \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1})\right). \tag{10}$$

It is clear from definition that $\lambda_1 \le c_1 \le \ldots \le c_n$.

Proposition 3.7. *Let $r \in \mathbb{N}$. $\mathcal{F}$ is $(\mathbf{a}, r)$-completable if and only if*

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) = c_r \;\; for \; r < n \tag{11}$$

or

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \ge c_n \;\; for \; r \ge n. \tag{12}$$

Moreover, if $c_r = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$ for some $r < n$, then $c_r = \lambda_1$.

Proof. Assume that $\mathcal{F}$ is $(\mathbf{a}, r)$-completable. If $r < n$ note that, by (9) in Corollary 3.5, we have $\lambda_1 = c_0 \le \ldots \le c_r = \lambda_1$ and $\lambda_1 = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$, so (11) holds. If $r \ge n$ then $\min\{n, r\} = n$ and Theorem 3.3 together with the definition of c_n imply that

$$\frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \ge c_n.$$

So in this case (12) holds. Conversely, if we assume (12), then it is clear $\mathcal{F}$ is $(\mathbf{a}, r)$-completable, by Theorem 3.3. Assume now that for some $r < n$, $c_r = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$. We show that $\mathcal{F}$ is $(\mathbf{a}, r)$-completable; indeed, since $nc_r = \sum_{i=1}^{r} a_i + \alpha$, then

$$rc_r + (n-r)c_r - \sum_{i=1}^{n-r}\lambda_i = \sum_{i=1}^{r} a_i + \sum_{i=1}^{r}\lambda_{n-i+1}$$

so by definition of c_r we have

$$\sum_{i=1}^{r}(a_i + \lambda_{n-i+1}) \le r\, c_r = \sum_{i=1}^{r}(a_i + \lambda_{n-i+1}) - \sum_{i=1}^{n-r}(c_r - \lambda_i) \le \sum_{i=1}^{r}(a_i + \lambda_{n-i+1}).$$

But then

$$\lambda_i = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right) \text{ for } 1 \le i \le n-r$$

and

$$\lambda_1 \ge \max_{1\le k\le r} \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1}),$$

so $\mathcal{F}$ is $(\mathbf{a}, r)$-completable, by Corollary 3.5. The last claim of the proposition is clear from our previous computations. □

We are now able to give a formula for the minimum $r \in \mathbb{N}$ such that $\mathcal{F}$ is $(\mathbf{a}, r)$-completable, when such an $r \in \mathbb{N}$ exists.

Theorem 3.8. *Let $\mathcal{F}$ be a $(\mathbf{a}, r)$-completable for some $r \in \mathbb{N}$. Let $r_0 \in \mathbb{N}$ be the minimum such that $\mathcal{F}$ is $(\mathbf{a}, r_0)$-completable. Then*

Case 1: *$r_0 < n$ if and only if $c_{r_0} = \frac{1}{n}\left(\sum_{i=1}^{r_0} a_i + \alpha\right)$*
Case2: *$r_0 \ge n$ if and only if $c_k \ne \frac{1}{n}\left(\sum_{i=1}^{k} a_i + \alpha\right)$ for all $1 \le k \le n-1$ and $r_0 \in \mathbb{N}$ is the minimum such that $c_n \le \frac{1}{n}\left(\sum_{i=1}^{r_0} a_i + \alpha\right)$.*

Proof. Note that, by Proposition 3.7, at least one the cases has to be fulfilled by some $r \in \mathbb{N}$. If we assume that case 1 holds for some $r < n$ then, by Proposition 3.7, $\mathcal{F}$ is $(\mathbf{a}, r)$-completable. By Corollary 3.6 case 1 does not hold for $k < n$ with $r \ne k$. It is clear that in this case $r_0 = r$.

Assume now that there is no $r < n$ satisfying case 1 above. Then, there exists $r \in \mathbb{N}$ such that $c_n \le \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$; by Proposition 3.7 we see that $\mathcal{F}$ is $(\mathbf{a}, r)$-completable. It is clear that r_0 is the minimum natural number r satisfying this condition. Finally note that if $r \in \mathbb{N}$ is such that $c_n \le \frac{1}{n}(\sum_{i=1}^{r} a_i + \alpha)$ then

$$\frac{1}{n}(\sum_{i=1}^{n} a_i + \alpha) \le c_n \le \frac{1}{n}(\sum_{i=1}^{r} a_i + \alpha) \ \Rightarrow \sum_{i=1}^{n} a_i \le \sum_{i=1}^{r} a_i$$

and $r \ge n$ since for every $i \in \mathbb{N}$, $a_i > 0$. □

The next example shows that it is possible to obtain a set of vectors $\mathcal{F}$ and a sequence $\mathbf{a}$ such that $\mathcal{F}$ is $(\mathbf{a}, r)$-completable for only one $r \in \mathbb{N}$ (in virtue of Corollary 3.6, $r < n$).

Example 3.9. Let $\mathcal{F} = \{\sqrt{2}e_1, \sqrt{2}e_2, e_3\}$ in $\mathbb{C}^3$ where $\{e_i\}$ is the canonical orthonormal basis and let $\mathbf{a} = \{\left(\frac{1}{4}\right)^{i-1}\}_{i=1}^{\infty}$. Then, easy computations show that the eigenvalues of $S^{\mathcal{F}}$ are $\lambda_1 = 2$, $\lambda_2 = 2$ and $\lambda_3 = 1$, so $\alpha = \operatorname{tr} S^{\mathcal{F}} = 5$. By Corollary 3.5 $\mathcal{F}$ is $(\mathbf{a}, 1)$-completable since $\lambda_1 = \frac{1}{3}(a_1 + \alpha)$ and $\lambda_1 \ge a_1 + \lambda_3$. Moreover, it is clear that if we add the vector e_3 to $\mathcal{F}$ we obtain a 2-tight frame.

On the other hand, it easy to see that $\frac{1}{3}(\sum_{i=1}^{\infty} a_i + \alpha) = \frac{19}{9} < \frac{17}{8} = c_3$ so, by Proposition 3.7, $\mathcal{F}$ is not $(\mathbf{a}, r)$-completable for any $r \ge 3$.

In fact, as the following proposition shows, if $\mathcal{F}$ is $(\mathbf{a}, r)$-completable with $r < n$, the existence of some $r_1 \geq n$ such that $\mathcal{F}$ is $(\mathbf{a}, r_1)$-completable depends only on the tail of the sequence, $\{a_i\}_{i=r+1}^{\infty}$.

Proposition 3.10. *Let $\mathcal{F}$ be $(\mathbf{a}, r)$-completable for some $r < n$. There exists $r_1 \in \mathbb{N}$ with $r_1 \geq n$ and such that $\mathcal{F}$ is $(\mathbf{a}, r_1)$-completable if and only if*

$$\frac{1}{n} \sum_{i=r+1}^{r_1} a_i \geq \max_{r+1 \leq k \leq n} \frac{1}{k} \sum_{i=r+1}^{k} a_i .$$

Proof. By Theorem 3.3, $\mathcal{F}$ is $(\mathbf{a}, r_1)$-completable if and only if

$$\frac{1}{n}\left(\sum_{i=1}^{r_1} a_i + \alpha\right) \geq \lambda_1 \text{ and } \frac{1}{n}\left(\sum_{i=1}^{r_1} a_i + \alpha\right) \geq \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1}),\ 1 \leq k \leq n.$$

By hypothesis and Corollary 3.5,

$$\lambda_i = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right),\ 1 \leq i \leq n - r \text{ and } \lambda_1 \geq \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1}),\ 1 \leq k \leq r$$

since $\mathcal{F}$ is $(\mathbf{a}, r)$-completable with $r < n$. So $\mathcal{F}$ is $(\mathbf{a}, r_1)$-completable if and only if

$$\frac{1}{n}\left(\sum_{i=1}^{r_1} a_i + \alpha\right) \geq \frac{1}{k}\sum_{i=1}^{k}(a_i + \lambda_{n-i+1}),\ r+1 \leq k \leq n$$

or equivalently, if for every $r + 1 \leq k \leq n$

$$\begin{aligned}
\sum_{i=1}^{r} a_i + \alpha + \sum_{i=r+1}^{r_1} a_i &\geq \frac{n}{k}\left(\sum_{i=1}^{r} a_i + \sum_{i=r+1}^{k} a_i + \alpha - (n-k)\lambda_1\right) \\
\sum_{i=1}^{r} a_i + \alpha + \sum_{i=r+1}^{r_1} a_i &\geq \frac{n}{k}\left(\sum_{i=1}^{r} a_i + \alpha\right) + \frac{n}{k}\sum_{i=r+1}^{k} a_i - \frac{n-k}{k}\left(\sum_{i=1}^{r} a_i + \alpha\right) \\
\sum_{i=1}^{r} a_i + \alpha + \sum_{i=r+1}^{r_1} a_i &\geq \sum_{i=1}^{r} a_i + \alpha + \frac{n}{k}\sum_{i=r+1}^{k} a_i \\
\sum_{i=r+1}^{r_1} a_i &\geq \frac{n}{k}\sum_{i=r+1}^{k} a_i \ ,
\end{aligned}$$

since by hypothesis $\lambda_i = \frac{1}{n}(\sum_{i=1}^{r} a_i + \alpha)$ for $1 \leq i \leq n - r$. □

3.2. **Completing with a infinite number of vectors. Proof of Theorems A and B.** In this section we consider some complementary results to those obtained in the previous section and prove Theorems A and B.

If $\mathcal{F} = \{f_i\}_{i=1}^{p}$ and $\mathbf{a}$ are as before, then a necessary condition for $\mathcal{F}$ to be $(\mathbf{a}, \infty)$-completable is that $\mathbf{a} \in \ell^1(\mathbb{N})$ (Remark 3.2).

Theorem 3.11. *$\mathcal{F}$ is $(\mathbf{a},\infty)$-completable (by a Bessel sequence) if and only if $\mathbf{a}\in\ell^1(\mathbb{N})$, $\frac{1}{n}\left(\sum_{i=1}^\infty a_i+\alpha\right)\geq\lambda_1$ and*

$$\frac{1}{n}\left(\sum_{i=1}^{\infty} a_i+\alpha\right)\geq\frac{1}{k}\sum_{i=1}^{k}(a_i+\lambda_{n-i+1}),\ 1\leq k\leq n, \tag{13}$$

or equivalently if $(\mathbf{a}\in\ell^1(\mathbb{N}))$

$$\frac{1}{n}\left(\sum_{i=1}^{\infty} a_i+\alpha\right)\geq c_n. \tag{14}$$

The proof of Theorem 3.11, which is based on Theorem 2.3, is similar to that of Theorem 3.3 and Proposition 3.7; we leave the details to the interested reader.

Proof of Theorem (A). The first part of the theorem is Theorem 3.3, while the second part is Theorem 3.11. □

Proof of Theorem (B). Assume there exists a natural number $r\in\mathbb{N}$ such that $\mathcal{F}$ is $(\mathbf{a},r)$-completable. Then $r_0\leq r$ and in this case the theorem follows from Theorem 3.8. If there is no $r\in\mathbb{N}$ such that $\mathcal{F}$ is $(\mathbf{a},r)$-completable, then $\mathcal{F}$ is $(\mathbf{a},\infty)$-completable so by Theorem 3.11 $\mathbf{a}\in\ell^1(\mathbb{N})$ and $\frac{1}{n}(\sum_{i=1}^\infty a_i+\alpha)\geq c_n$. If $\frac{1}{n}(\sum_{i=1}^\infty a_i+\alpha)>c_n$ then there exists $r\in\mathbb{N}$ such that $\frac{1}{n}(\sum_{i=1}^r a_i+\alpha)\geq c_n$ but then, by Proposition 3.8 we get that $\mathcal{F}$ is $(\mathbf{a},r)$-completable, a contradiction. □

We finish with the counter-part of Proposition 3.10 for the infinite completion case.

Proposition 3.12. *Let $\mathbf{a}\in\ell^1(\mathbb{N})$ and let $\mathcal{F}$ be $(\mathbf{a},r)$-completable for some $r<n$. Then, $\mathcal{F}$ is $(\mathbf{a},\infty)$-completable if and only if*

$$\frac{1}{n}\sum_{i=r+1}^{\infty} a_i\geq\max_{r+1\leq k\leq n}\frac{1}{k}\sum_{i=r+1}^{k} a_i.$$

4. Equal norm tight frames

In this section we consider the particular case when $\mathbf{a}=\{a_i\}_{i\in\mathbb{N}}$ is a constant sequence, $a_i=1$ for all $i\in\mathbb{N}$ (the general case follows in an analogous way). Note that in this case $\mathcal{F}$ is $(\mathbf{a},r)$-completable for some $r\in\mathbb{N}$; so we shall compute the minimum natural number r of vectors with norm one we have to add to $\mathcal{F}$ in order to get a tight frame. We keep the notation of the previous section for $\mathcal{F}=\{f_i\}_{i=1}^p$, $S^{\mathcal{F}}$, $\lambda_1\geq\ldots\geq\lambda_n$ and α.

Remark 4.1. Under our present assumption that $a_i=1$ for all $i\in\mathbb{N}$ we have that

$$c_r=\max\left(\lambda_1, 1+\frac{1}{r}\sum_{i=1}^{r}\lambda_{n-i+1}\right).$$

Indeed, if $j\leq k$ then $\frac{1}{j}\sum_{i=1}^j\lambda_{n-i+1}\leq\frac{1}{k}\sum_{i=1}^k\lambda_{n-i+1}$ since $\lambda_1\geq\ldots\geq\lambda_n\geq 0$.

This simpler formula for the coefficients c_j provides the following characterization for the optimal number of elements for tight completions with norm one vectors.

Theorem 4.2. *Let* $h := \sum_{i=2}^{n} \lambda_1 - \lambda_i$*, and denote by* r_0 *the minimum number of norm one vectors we have to add to* $\mathcal{F}$ *in order to have a tight frame.*

Case 1: *Suppose that* $h < n$*. Then* $r_0 = h$ *if* $h \in \mathbb{N}$ *and* $1 + \frac{1}{h}\sum_{i=1}^{h} \lambda_{n-i+1} \leq \lambda_1$ *(in particular,* $c_h = \lambda_1$*). Otherwise,* $r_0 = n$*.*

Case 2: *If* $h \geq n$ *,* r_0 *is the minimum integer greater than or equal to* h*.*

Proof. Assume that $h < n$; then, since $h = n\lambda_1 - \alpha$, we have that $c_n = 1 + \frac{\alpha}{n}$ by Remark 4.1. If in addition $h \in \mathbb{N}$ and $1 + \frac{1}{h}\sum_{i=1}^{h} \lambda_{n-i+1} \leq \lambda_1$, so $c_h = \frac{1}{n}(h + \alpha) = \lambda_1$, then $r_0 = h$ by Theorem 3.8. Otherwise, $c_k \neq \frac{1}{n}(k + \alpha)$ for all $k < n$ (if $c_k = \frac{1}{n}(k + \alpha)$ for some $k < n$, then by Proposition 3.7 $c_k = \lambda_1$ and h would be a natural number); since $c_n = 1 + \frac{\alpha}{n}$, the minimum integer greater than or equal to $nc_n - \alpha$ is n so $r_0 = n$ by Theorem 3.8.

Finally, $h \geq n$ implies that $c_k \neq \frac{1}{n}(k + \alpha)$ for all $k < n$ and $c_n = \lambda_1$. Therefore, again by Theorem 3.8, r_0 is the minimum integer greater than or equal to $n\lambda_1 - \alpha = h$. $\square$

Remark 4.3. Note that, as a consequence of Theorem 4.2, if $\lambda_1 - \lambda_2 \geq \frac{n}{n-1}$ then $h = \sum_{i=2}^{n} \lambda_1 - \lambda_i \geq n$, so $\mathcal{F}$ can not be completed to a tight frame with less than n unit norm vectors.

In addition, the number h can be seen as a kind of measure of how "far of being tight" is the set of vectors $\mathcal{F}$, in the sense that $h = 0$ if and only if the set $\mathcal{F}$ is already a tight frame.

Example 4.4. This example is taken from [4]. It is interesting because it shows the difference between the cases when we can complete $\mathcal{F}$ to a tight frame with $r < n$ or $r \geq n$ vectors. Let $f_1 = (1, 0)$ and $f_2 = (\cos\theta, \sin\theta)$ in $\mathbb{R}^2$, and consider $a_i = 1\ \forall i$. It easy to see that the eigenvalues of $S^{\mathcal{F}}$ are $1 \pm \cos\theta$, hence $h = \lambda_1 - \lambda_2 = 2\,|\cos\theta\,|$. Therefore, by Theorem 4.2, the minimum number r_0 of unit vectors we have to add to obtain a tight frame is 2, unless $\theta = \frac{2}{3}\pi$ or $\theta = \frac{4}{3}\pi$, where $r_0 = 1$. Note that when $r_0 = 1$ the tight frame obtained is the well known "Mercedes Benz" (it is –up to rigid rotations, reflections and negation of individual vectors– the only unit norm tight frame on $\mathbb{R}^2$ with three elements [5]).

A consequence of Theorem 4.2 is the characterization of the minimum number of vectors that we have to add in order to get a tight frame, in the particular case when $\mathcal{F}$ is a unit norm tight frame on its linear span.

Proposition 4.5. *Let* $\mathcal{F} = \{f_i\}_{i=1}^{p}$ *be a unit norm* $\frac{p}{d}$*-tight frame on its span, where* $d < n$ *is the dimension of span* $\mathcal{F}$*. Then, the minimum number* r_0 *of unit norm vectors we have to add to* $\mathcal{F}$ *in order to obtain a tight frame in* $\mathcal{H}$ *is:*

a) $(n-d)\frac{p}{d}$ *if* $(n-d)\frac{p}{d} < n$ *and* $(n-d)\frac{p}{d} \in \mathbb{N}$*.*
b) n *if* $(n-d)\frac{p}{d} < n$ *and* $(n-d)\frac{p}{d} \notin \mathbb{N}$*.*
c) *the minimum integer greater than or equal to* $(n-d)\frac{p}{d}$ *if* $(n-d)\frac{p}{d} \geq n$*.*

Proof. Since $\mathcal{F}$ is an unit norm tight frame on a subspace of dimension d, the eigenvalues of $S^{\mathcal{F}}$ are: $\lambda_i = \frac{p}{d} \geq 1$ for $1 \leq i \leq d$, and $\lambda_i = 0$ for $d+1 \leq i \leq n$. Therefore, $h = \sum_{i=2}^{n} \lambda_1 - \lambda_i = (n-d)\frac{p}{d}$. Moreover, if $h < n$ and $h \in \mathbb{N}$, then $1 + \frac{1}{h}\sum_{i=1}^{h} \lambda_{n-i+1} = \lambda_1$. Indeed,

$$1 + \frac{1}{h}\sum_{i=1}^{h} \lambda_{n-i+1} = 1 + \frac{h-(n-d)}{h}\frac{p}{d} = \frac{p}{d} \tag{15}$$

the proposition is then a consequence of Theorem 4.2. □

5. Some remarks regarding algorithmic implementation

Let $\mathcal{F} = \{f_i\}_{i=1}^{p} \subseteq \mathcal{H}$ and assume that $\mathbf{a}$ is a divergent sequence. Then, by Remark 3.4, $\mathcal{F}$ is $(\mathbf{a}, r)$-completable for some $r \in \mathbb{N}$. From the proof of Theorem 3.3 we see that, if $c = \frac{1}{n}\left(\sum_{i=1}^{r} a_i + \alpha\right)$ then equation (8) holds. Therefore, by Theorem 2.3, *theoretically*, there exists a Bessel sequence $\mathcal{G} = \{g_i\}_{i=1}^{r} \subset \mathcal{H}$ such that $\|g_i\|^2 = a_i$ for $1 \leq i \leq r$ and $S^{\mathcal{G}} = cI - S^{\mathcal{F}}$. In this case, $\mathcal{G}$ is a $(\mathbf{a}, r)$-completion of $\mathcal{F}$; moreover, if $r \in \mathbb{N}$ is obtained as in Theorem 3.8 then $\mathcal{G}$ would be $(\mathbf{a}, r)$-tight completion having the minimum number of vectors for which a tight completion of $\mathcal{F}$ exists. Although constructive, the proof of Theorem 2.3 is not practicable; it depends on some matrix decompositions which can not be performed efficiently by a computer for large values of $t = \min\{n, r\}$.

There are several recent papers related to algorithmic construction of frames with additional properties. In [2] Casazza and Leon considered the problem of constructing frames with prescribed properties from an algorithmic point of view; in particular, they obtained an algorithm for constructing tight frames with prescribed norms of its elements, under the admissibility conditions of Theorem 2.3. In [4] there is a fast algorithm for constructing tight frames with prescribed norms of its elements based on Householder transformations; in [7] a fast algorithmic proof of some results related to the Schur-Horn theorem is considered and as a consequence a generalized one-sided Bendel-Mickey algorithm (see Theorem 5.1 below) is obtained. Still, as far as we know, the problem of constructing a frame for $\mathcal{H}$ with prescribed general (positive definite) frame operator and norms (that are admissible in the sense of Theorem 2.3) using an *efficient* computable algorithm has not been solved: we remark that for the purposes of this discussion, the diagonalization of a positive semi-definite matrix is considered as *not* efficiently computable. If such an algorithm is obtained, then optimal tight frame completions can be constructed as described in the first paragraph of this section. In what follows we shall consider a not so optimal tight frame completion of a given set $\mathcal{F} = \{f_i\}_{i=1}^{p}$ but that is efficiently algorithmic computable, based on the generalized one-sided Bendel-Mickey algorithm and the Cholesky's decomposition.

Let us begin with the following result from [7]. We remark that our notation is opposite to that in [7] so we translate their result into our terminology.

Theorem 5.1 ([7]). *Let* $\mathbf{a} = \{a_i\}_{i=1}^r$, $\mathbf{b} = \{b_i\}_{i=1}^r$ *be two finite and non-increasing sequences of positive numbers such that* $\mathbf{a} \prec \mathbf{b}$. *Let* X *be an* $n \times r$ *matrix whose squared columns norms are listed by* $\mathbf{b}$. *Then there is a finite sequence of algorithmic computable plane rotations* $U_1, \ldots, U_{r-1} \in \mathbb{M}_r(\mathbb{C})$ *such that* $X(U_1 \cdots U_{r-1})$ *has squared columns norms listed by* $\mathbf{a}$.

Actually, each plane rotation that appears in the theorem above operates non-trivially in the coordinate plane $\operatorname{span}\{e_i, e_j\}$ for some $1 \le i, j \le r$ (see [7] for details). Note that the initial matrix X and the final matrix $Y = X(U_1 \cdots U_{r-1})$ satisfy $XX^* = YY^*$.

Taking into account Theorem 5.1, an strategy to construct a frame with prescribed frame operator $S \in \mathbb{M}_n(\mathbb{C})$ and norms of its elements listed by $\mathbf{a}$ (satisfying the conditions in Theorem 2.3) would be the following: consider a diagonalization $S = U\operatorname{diag}(\lambda_1, \ldots, \lambda_n)U^*$ and the factorization $XX^* = S$ with $X = U\operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Note that the squared norms of the columns of X are listed by $(\lambda_1, \ldots, \lambda_n)$ so we can apply Theorem 5.1 and obtain $Y = X(U_1 \cdots U_{r-1})$ with $YY^* = S$ with the squared norms of the columns of Y given by $\mathbf{a}$. Unfortunately, we consider this procedure as not an efficiently computable one, so we have to find an alternative approach.

Remark 5.2. In what follows we shall make use of the well known Cholesky's decomposition $S = RR^*$ for a positive definite matrix S. Note that in this case Cholesky's decomposition is unique, and there are several strategies for calculating the matrix R is an efficient way.

5.1. **Algorithm for constructing tight completions.** Along this section we prove Theorem C; we begin with an informal discussion of the algorithm.

Assume that the non-increasing sequence of positive numbers $\{a_i\}_{i=1}^\infty$ forms a divergent series, so that $\mathcal{F}$ is $(\mathbf{a}, t)$-completable for some $t \in \mathbb{N}$. Let $S = S^{\mathcal{F}}$ and let $c > \|S\|$ that we shall consider as a variable. We will obtain an algorithmic computable value of c for which the Cholesky's decomposition $cI_n - S = RR^*$ satisfies that the squared norms of the columns of R mayorize $\{a_i\}_{i=1}^r$ for an integer $r \ge n$. Once we have obtained such c, we apply Theorem 5.1 and get a finite sequence $\mathcal{G} = \{g_i\}_{i=1}^r$ with frame operator $cI_n - S$ and $\|g_i\|^2 = a_i$, for $1 \le i \le r$.

Let $c \ge \|S\| + \beta$ so $\lambda_{\min}(cI - S) = c - \|S\| \ge \beta$, where $\beta > 0$ is a fixed number controlling the invertibility of $cI - S$. Let $R = R(c)$ be the upper triangular matrix obtained from the Cholesky's decomposition of $cI - S$ (note that the hypothesis on c is made in order that the Cholesky's algorithm becames stable). Then $RR^* = cI - S$ and note that $c - \|S\| = \lambda_{\min}(RR^*) = \lambda_{\min}(R^*R)$ so, if $C_i(R)$ denotes the i-th column of R then

$$\min_{1 \le i \le n} \|C_i(R)\|^2 \ge c - \|S\|,$$

since $\|C_i(R)\|^2 = (R^*R)_{ii}$ and $(R^*R)_{ii} \ge \lambda_{\min}(R^*R)$ for $1 \le i \le n$. In particular $\sum_{i=1}^k \|C_i(R)\|^2 = \sum_{i=1}^k (R^*R)_{ii} \ge k \cdot (c - \|S\|)$. Let $c \ge \max(\|S\| + \beta, \|S\| + a_1)$

and note that then

$$c \geq \frac{1}{k}\sum_{i=1}^{k} a_i + \|S\|, \quad \text{for } 1 \leq k \leq n \tag{16}$$

since $\frac{1}{k}\sum_{i=1}^{k} a_i \geq \frac{1}{h}\sum_{i=1}^{h} a_i$ if $1 \leq k \leq h \leq n$. Let $r \in \mathbb{N}$ be such that

$$\sum_{i=1}^{r-1} a_i < \sum_{i=1}^{n} \|C_i(R(c))\|^2 = c \cdot n - \operatorname{tr}(S) \leq \sum_{i=1}^{r} a_i \tag{17}$$

so $r \geq n$. We define $c' = \frac{1}{n}(\sum_{i=1}^{r} a_i + \operatorname{tr}(S^{\mathcal{F}}))$, where r is defined by (17) so that, if $R(c')$ denotes the Cholesky's decomposition of $c'I - S^{\mathcal{F}}$ then we get $(a_i)_{i=1}^{r} \prec (\|C_i(R(c'))\|^2)_{i=1}^{n}$.

Thus, with this $c' \in \mathbb{R}$ and $r \in \mathbb{N}$ we can apply Theorem 5.1 to the matrix $X = [R(c'),\ 0_{n\times(r-n)}]$ and get the (efficiently algorithmic computable) $n \times r$ matrix Y such that $YY^* = S$ and $\|C_i(Y)\|^2 = a_i$ for $1 \leq i \leq r$; setting $g_i = C_i(Y)$ we get $\{g_i\}_{i=1}^{r}$ with the desired properties. We briefly resume the previous considerations in the following pseudo-code implementation:

- Find an algorithmic computable upper bound d for $\|S\|$.
- Compute $c = \max(d + \beta, d + a_1)$ (where $\beta > 0$ is previously fixed) and $r \in \mathbb{N}$ satisfying (17).
- Redefine $c := \frac{1}{n}(\sum_{i=1}^{r} a_i + \operatorname{tr}(S^{\mathcal{F}}))$.
- Compute the Cholesky's decomposition $cI - S = RR^*$.
- Apply Theorem 5.1 to the $n \times r$ matrix $[R,\ 0_{n\times(r-n)}]$ and get the $n \times r$ matrix Y such that $cI - S = YY^*$ and $\|C_i(Y)\|^2 = a_i$ for $1 \leq i \leq r$.
- Define $g_i = C_i(Y)$ for $1 \leq i \leq r$.

Example 5.3. Assume that $\|f_i\| = 1$ for $1 \leq i \leq p$ and that $\|a_i\| = 1$, so we are looking for unit norm tight completions of a unit norm family of vectors $\mathcal{F}$. In this case, it is shown in [4] that if $d = [\![\|S^{\mathcal{F}}\| + 1]\!]$, where $[\![h]\!]$ denotes the smallest integer greater than or equal to h, there always exists a unit norm tight completion of $\mathcal{F}$ with $dn - p$ elements. Our arguments above show that there exists an efficiently algorithmic computable unit norm tight completion with $[\![n \cdot (\|S^{\mathcal{F}}\| + 1) - p]\!]$ (assuming that we can compute efficiently $\|S^{\mathcal{F}}\|$ and seting $\beta = 1$). Note that in general we have that $n \cdot [\![\|S^{\mathcal{F}}\| + 1]\!] - p \geq [\![n \cdot (\|S^{\mathcal{F}}\| + 1) - p]\!]$.

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